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A Generalized Biharmonic Equation and Its Applications to Double- Diffusive Convection Coupled With Cross-Diffusions

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ABSTRACT: The equation $(A\nabla^4 + B\nabla^2 + C)\ \chi = \mathbf{0}$ with matrix coefficients A, B, C is studied for homogeneous boundary conditions. An integral constraint is derived for the above system leading to a relationship between the matrix involved. As a consequence, results are obtained for double-diffusive convection problem coupled with cross-diffusions for Veronis' and Stern's type configurations.
KEYWORDS: Double-diffusive convection; Dufour and Soret effects; Rayleigh numbers; Prandtl number

I. INTRODUCTION

The stability properties of binary fluids are quite different from pure fluids because of Soret and Dufour effects [1, 2]. An externally imposed temperature gradient produces a chemical potential gradient and the phenomenon known as the Soret effect, arises when the mass flux contains a term that depends upon the temperature gradient. The analogous effect that arises from a concentration gradient dependent term in the heat flux is called the Dufour effect. It is now well established fact that the thermosolutal and Soret-Dufour problems are quite closely related, in fact, they are formally identical and identification is done by means of a linear transformation that takes the equations and boundary conditions for the latter problem into those for the former. The analysis of double diffusive convection becomes complicated in case when the diffusivity of one property is much greater than the other. Further, when two transport processes take place simultaneously, they interfere with each other and produce cross diffusion effect (Dufour-Soret effects). The Soret and Dufour coefficients describe the flux of mass caused by temperature gradient and the flux of heat caused by concentration gradient respectively. The coupling of the fluxes of the stratifying agents is a prevalent feature in multicomponent fluid systems. In general, the stability of such systems is also affected by the cross-diffusion terms. Generally, it is assumed that the effect of cross diffusions on the stability criteria is negligible. However, there are liquid mixtures for which cross diffusions are of the same order of magnitude as the diffusivities. There are only few studies available on the effect of cross diffusion on double diffusion convection largely because of the complexity in determining these coefficients. The effect of Soret coefficient on the double–diffusive convection has been studied by [3]. They have reported that the magnitude and sign of the Soret coefficient were changed by varying the composition of the mixture. The problem of Dufour-driven thermosolutal convection has also been considered by [4] and results concerning the linear growth rate and behavior of oscillatory motions have been established.

Bounds to eigenvalues of ordinary homogeneous system are a problem of interest for their own sake and assume added significance when these systems represent physical situations and the eigenvalues are not exactly obtainable [Warren [5]]. The method of quadratic forms is a familiar device which often succeeds in characterizing the eigenvalues and establishing the bounds, and an important application of this occurs in a certain class of hydrodynamics stability problems [see, for example, Chandrasekhar [6] and Lin [7].There is a basic similarity of approach in all these cases ‚namely." Multiplying the governing equation by the conjugate eigenfunction and integrating the resulting equation over the range of the boundary conditions,"

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but apart from this each case is treated as a particular problem and the various other steps taken in any two cases often appear ad hoc and unrelated. The question that naturally emerges is the following: Can a unified mathematical treatment be given to the above class of problems wherein the basic content of the method of quadratic forms is retained while its ad hoc nature (in the context of the above mentioned applications) is removed.

In the present paper we show that a generalized biharmonic equation with matrix coefficients and homogeneous boundary conditions provides the basis for the unified approach for a subclass of the above class of problems such that in any two cases the application differs only in the choice of the matrices involved. As a consequence, known as well as unknown results are obtained for double-diffusive convection coupled with cross-diffusions in case of Veronis' and Stern's type configurations[8,9].

II. FORMULATION AND RESULTS

Consider the coupled system of n linear homogeneous partial differential equations

$$[A\nabla^4 + B\nabla^2 + C(x)]\chi(x) = 0, \qquad (1)$$

in a simply connected open subset V of the Euclidean space R^m; x refers to the point $(x_1, x_2, \ldots\ldots x_m)$ of V; A and B are $n \times n$ matrices with complex constant entries; $C(x)$ is an $n \times n$ matrix with complex valued functions on V as entries; ∇^{2k} stands for the operator $\{\sum_{i=1}^{m}(\partial^2/\partial x_i^2)\}^k, k = 1, 2$; $\chi(x)$ is the column vector $\{\chi_i(x)\}_{n\times1}$, $\chi_i(x)$ being complex valued functions on V and $\nabla^{2k}\chi(x) \equiv \{\nabla^{2k}\chi_i(x)\}_{n\times1}$.

We consider equation (1) together with homogeneous boundary conditions

$$\chi(x) = 0 \text{ and either } A\frac{\partial\chi(x)}{\partial n} = 0 \text{ or } A\nabla^2\chi(x) = 0, \qquad (2)$$

on the boundary S of V. Here $A(\partial\chi/\partial n)$ and $A\nabla^2\chi$ stand for the vectors $\{\sum_{j=1}^{n} A_{ij}(\partial x_j/\partial n)\}_{n\times1}$ and $\{\sum_{j=1}^{n} A_{ij}\nabla^2\chi_j\}_{n\times1}$, respectively, and $\partial/\partial n$ denotes the differentiation along the positive outward drawn normal at any point on S. Henceforth, we shall use the summation convection for repeated indices unless stated otherwise.

In Section 3, a necessary condition for the existence of a non-trivial solution $\chi(x)$ of equations (1) and (2) is obtained in the form of an integral relation involving the solution, which is then specialized to a form more suited to applications. The integral relation yields an inequality between the eigenvalues of the matrices involved and has no explicit dependence on the solution.

In Section 4, the above results are shown to lead to the following consequences in the field of hydrodynamic stability:

That the complex growth rate $p(= p_r + ip_i)$ of an arbitrary oscillatory perturbation, neutral or unstable, in the linear stability problem of thermohaline convection (Veronis′/Stern's configuration) with dynamically free or rigid boundaries, must lie in an open disk in the right half of the $p_r p_i$-plane with centre as origin and $(\text{radius})^2 = R'_S\sigma/-R'_T\sigma$,

where R'_S is the modified salinity Rayleigh number, R'_T is the modified thermal Rayliegh number and σ is the Prandtl number (Banerjee et al. [10]).

III. MATHEMATICAL ANALYSIS

Lemma 1: If a solution χ satisfying equations (1) and (2) exists then we have

$$\int_V (\nabla^2\chi)^\dagger A_1(\nabla^2\chi)dV - \int_V (\text{grad}\,\chi)^\dagger B_1(\text{grad}\,\chi)dV + \int_V \chi^\dagger C_1(x)\chi dV = 0, \qquad (3)$$

where

$$A_1 = \frac{A - A^\dagger}{2i}, \quad B_1 = \frac{B - B^\dagger}{2i}, \quad C_1(x) = \frac{C(x) - C^\dagger(x)}{2i},$$

$\text{grad}\,\chi = \{\text{grad}\,\chi_i\}_{n\times1}$ and the grad operator is in R^m, and the symbols † and * stand for the complex conjugate transpose and complex conjugate, respectively, so that the second term on the left hand side of equation (3), for example, stands for $\int_V (\partial\chi_{ii}/\partial x_j)^*(B_1)_{ik}(\partial\chi_k/\partial x_j)dV$.

Proof: Multiplication of equation (1) to the left by $\chi^\dagger$ and integration over the domain V gives

$$\int_V \chi^\dagger A\nabla^4\chi dV + \int_V \chi^\dagger B\nabla^2\chi dV + \int_V \chi^\dagger C(x)\chi dV = 0. \qquad (4)$$

Now,

$$\int_V \chi^\dagger A\nabla^4\chi dV = \int_V \chi_i^* A_{ij}\nabla^4\chi_j dV = A_{ij}\int_V \chi_i^*\nabla^2(\nabla^2\chi_j)dV, \qquad (5)$$

and making repeated use of Gauss′ theorem in R^m and boundary conditions (2), we get

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$$\begin{aligned}A_{ij}\int_V \chi_i^*\nabla^2(\nabla^2\chi_j)dV &= A_{ij}\int_S \chi_i^*\frac{\partial}{\partial n}(\nabla^2\chi_j)ds - A_{ij}\int_V \text{grad}\,\chi_i^*.\text{grad}\,(\nabla^2\chi_j)dV\\ &= -A_{ij}\int_V \text{grad}\,\chi_i^*.\text{grad}\,(\nabla^2\chi_j)dV\\ &= -A_{ij}\int_S \nabla^2\chi_i\frac{\partial}{\partial n}(\chi_i^*)ds + A_{ij}\int_V \nabla^2\chi_i\nabla^2\chi_i^* dV\\ &= A_{ij}\int_V \nabla^2\chi_i\nabla^2\chi_i^* dV = \int_V (\nabla^2\chi)^\dagger A(\nabla^2\chi)dV\ , \qquad (6)\end{aligned}$$

the symbol "." is the dot product. Similarly, we have

$$\int_V \chi^\dagger B\nabla^2\chi dV = -\int_V (\text{grad}\,\chi)^\dagger B(\text{grad}\,\chi)dV\ . \qquad (7)$$

Using equations (5) – (7), we have from equation (4)

$$\int_V (\nabla^2\chi)^\dagger A(\nabla^2\chi)dV - \int_V (\text{grad}\,\chi)^\dagger B(\text{grad}\,\chi)dV + \int_V \chi^\dagger C(x)\chi dV = 0\ . \qquad (8)$$

The imaginary part of equation (8) gives

$$\int_V (\nabla^2\chi)^\dagger A_1(\nabla^2\chi)dV - \int_V (\text{grad}\,\chi)^\dagger B_1(\text{grad}\,\chi)dV + \int_V \chi^\dagger C_1(x)\chi dV = 0\ ,$$

and this proves the lemma.

Theorem 1: Under the hypothesis of Lemma 1, if$A_1 = lA_2$, $B_1 = lB_2$, $C_1(x) = -lC_2(x)$, where l is a non-zero real number; A_2 is a non-negative definite Hermitian matrix and B_2 is a positive definite Hermitian matrix, $C_2(x)$ is a Hermitian matrix; then

$$\text{Sup}[\text{Eigenvalues of } C_2(x)] > 0\ , \qquad (9)$$

"Sup" being taken over all the eigenvalues of $C_2(x)$ over all x in V.
Proof: Since A_2 is no

n-negative definite and B_2 positive definite, we have

$$\int_V (\nabla^2\chi)^\dagger A_2(\nabla^2\chi)dV + \int_V (\text{grad}\,\chi)^\dagger B_2(\text{grad}\,\chi)dV > 0.$$

Lemma 1 then gives

$$\int_V \chi^\dagger C_2(x)\chi dV > 0\ . \qquad (10)$$

Let $u_1(x), u_2(x), \dots\dots\dots, u_n(x)$ be the n eigenvectors of $C_2(x)$. Here it is to be carefully noted that for each $i = 1, 2, \dots\dots, n$ and each x in $V, u_i(x)$ is an $n\times 1$ matrix. Further, let $\lambda_i(x)$ be the eigenvalue of $C_2(x)$ corresponding to $u_i(x)$ so that $C_2(x)u_i(x) = \lambda_i(x)u_i(x)$ (no summation implied). The matrix $C_2(x)$ being Hermitian in a finite dimensional space, the spectral theorem gives that these form an orthonormal basis for R^n for each x in V, i.e. $u_i^\dagger u_j(x) = \delta_{ij}$. We thus have an expansion

$$\chi(x) = \sum_{i=1}^{n} f_i(x)u_i(x), \qquad (11)$$

where $f_i(x)$ are complex valued functions on V.
Substitution of $\chi(x)$ from equation (11) in the left hand member of inequality (10) yields

$$\begin{aligned}\int_V \chi^\dagger C_2(x)\chi dV &= \int_V \left[\sum_{i=1}^{n} f_i(x)u_i(x)\right]^\dagger C_2(x)\left[\sum_{i=1}^{n} f_i(x)u_i(x)\right]dV\\ &= \int_V \left[\sum_{i=1}^{n} f_i^*(x)u_i^\dagger(x)\right]\left[\sum_{i=1}^{n}\lambda_i(x)f_i(x)u_i(x)\right]dV\\ &= \int_V \left[\sum_{i=1}^{n}|f_i(x)|^2\lambda_i(x)\right]dV\\ &\le \left[\underset{1\le i\le n\ ,x\in V}{Sup\ Sup}(\lambda_i(x))\right]\int_V \sum_{i=1}^{n}|f_i(x)|^2\, dV.\end{aligned}$$

Thus,

$$\int_V \chi^{\dagger} C_2(x)\chi dV \le \left[\underset{1\le i\le n,\, x\in V}{Sup\ Sup}(\lambda_i(x))\right]\int_V \chi^{\dagger}\chi dV\ . \qquad (12)$$

The theorem now follows from inequalities (10) and (12).

Corollary 1: Under the hypothesis of Theorem 1, if $C_2(x) = C_3(x) - C_4(x)$, then

$$Sup\ [Eigenvalues\ of\ C_3(x)] > Inf\ [Eigenvalues\ of\ C_4(x)], \qquad (13)$$

"Sup" and "Inf" being taken as in inequality (9).

The proof readily follows from Theorem 1.

Corrolary 2: Under the hypothesis of Theorem 1, if $B_2 = B_3 - B_4$, $C_2(x) = C_5(x) - C_6(x)$ and there exists a positive definite Hermitian matrix $H(x)$ s.t

$$\int_V [(grad\ \chi)^{\dagger} B_4\ grad\ \chi + \chi^{\dagger} C_5(x)\chi] dV < \int_V \chi^{\dagger} H(x)\chi dV\ , \qquad (14)$$

Then,

$$Sup[Eigenvalues\ of\ H(x)] > Inf[Eigenvalues\ of\ C_6(x)]\ , \qquad (15)$$

where "Sup" and "Inf" are taken as in inequality (9) and B_3 , B_4 , $C_5(x), C_6(x)$ are Hermitian matrices with B_3 , B_4 and $C_5(x)$ being positive definite.

The proof readily follows from Theorem 1.

Corollary 3: If Eq. (1) is replaced by

$$\left[A\nabla^4 + B\nabla^2 + C(x) + i\sum_{j=1}^{m}\gamma_j \frac{\partial}{\partial x_j}\right]\chi = 0,$$

γ_j's beings real constants, then the conclusions of Lemma1 and Theorem 1 are still valid.

Proof: We exactly follow the proof of Lemma 1 and note that the only additional term that we now have to consider on the left hand side of Eq.(4) is

$i\int_V \chi^{\dagger}\left[\sum_{j=1}^{m}\gamma_j \frac{\partial\chi}{\partial x_j}\right]$dv.

Now ,

$$i\int_V \chi^{\dagger}\left[\sum_{j=1}^{m}\gamma_j \frac{\partial\chi}{\partial x_j}\right] dV = i\sum_{j=1}^{m}\int_V \gamma_j\, \chi_r^{*}\frac{\partial\chi_r}{\partial x_j} dV. \qquad (16)$$

Further, let $\chi_r = u_r + i\, v_r$, we then have

$$i\int_V \chi^{\dagger}\left[\sum_{J=1}^{m}\gamma_J \frac{\partial\chi}{\partial x_J}\right] dV =$$

$$i\sum_{j=1}^{m}\gamma_j \int_V (u_r - iv_r)\frac{\partial}{\partial x_j}(u_r + iv_r) dV$$

$$= i\sum_{j=1}^{m}\gamma_j \int_V \frac{1}{2}\frac{\partial}{\partial x_j}(u_r^2 + v_r^2) dV \quad - \sum_{j=1}^{m}\gamma_j \int_V \frac{\partial}{\partial x_j}\left(u_r\frac{\partial v_r}{\partial x_j} - v_r\frac{\partial u_r}{\partial x_j}\right) dV$$

$$= -\sum_{j=1}^{m}\gamma_j \int_V \frac{\partial}{\partial x_j}\left(u_r\frac{\partial v_r}{\partial x_j} - v_r\frac{\partial u_r}{\partial x_j}\right) dV,$$

which is purely real. Conclusion of Lemma 1 and Theorem 1 therefore remain unchanged. This proves the corollary.

Remark1: It is clear that Theorem1 and its consequences are valid in a much more general setting. For instance $\chi(x)$ could be a vector in an finite dimensional Hilbert space,A_2, B_2 and $C_2(x)$ compact linear Hermitian operators, A_2 being non-negative definite and B_2 being positive definite.

IV. APPLICATION TO HYDRODNAMIC STABILITY

(a) Stability of Double-Diffusive Convection Coupled with Cross-diffusions for Veronis' configuration

The governing equations and boundary conditions for this problem are as follows:

Following the usual steps of linear stability theory the non- dimensional linearized perturbation equations governing the thermosolutal convection problem coupled with cross-diffusion with slight change in notations are easily seen to given by (Neild [11], Krusin [12]).

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \frac{p}{\sigma}\right)w = R_T\, a^2\theta - R_s a^2\phi \qquad (17)$$

$$\left(D^2 - a^2 - p\right)\theta + D_T\left(D^2 - a^2\right)\phi = -w,$$

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(18)

$$\left(D^2 - a^2 - \frac{p}{\tau}\right)\phi + S_T (D^2 - a^2)\,\theta = -\frac{w}{\tau},$$

(19) together with the boundary condition

$w = 0 = \theta = \phi = Dw$ at z=0 and z=1 (both boundaries rigid) (20)

or

$w = 0 = \theta = \phi = D^2 w$ at z =0 and z =1 (both boundaries dynamically free) (21)

or $w = 0 = \theta = \phi = Dw$ at z =0

$w = 0 = \theta = \phi = D^2 w$ at z=1 .

(lower boundary rigid and upper boundary dynamically free). (22)

The meanings of symbols from physical point of view are as follows;
z is the vertical coordinate, d/dz is differentiation along the vertical direction, a^2 is square of horizontal wave number, $\sigma = \frac{\upsilon}{\kappa}$ is the thermal Prandtl number, $\tau = \frac{\eta_1}{\kappa}$ is the Lewis number, $R_T = \frac{g\alpha\beta_1 d^4}{\kappa\upsilon}$ is the thermal Rayleigh number, $R_S = \frac{g\alpha\beta_2 d^4}{\kappa\upsilon}$ is the concentration Rayleigh number, $D_T = \frac{\beta_2 D_f}{\beta_1 \kappa}$ is the Dufour number, $S_T = \frac{\beta_1 S_f}{\beta_2 \eta_1}$ is the Soret number, ϕ is the concentration, θ is the temperature, p is the complex growth rate and w is the vertical velocity.

In equations (17)–(22), z is real independent variable such that $0 \le z \le 1$, $D = \frac{d}{dz}$ is differentiation w.r.t z , a^2 is a constant, $\sigma > 0$ is a constant, $\tau > 0$ is a constant, R_T and R_S are positive constants for the Veronis' configuration and negative constants for Stern's configuration, $p = p_r + ip_i$ is complex constant in general such that p_r and p_i are real constants and as a consequence the dependent variables $w(z) = w_r(z) + iw_i(z)$, $\theta(z) = \theta_r(z) + i\theta_i(z)$ and $\phi(z) = \phi_r(z) + i\phi_i(z)$ are complex valued functions(and their real and imaginary parts are real valued).

We now introduce the transformations

$$\tilde{w} = (S_T + B)\,w \qquad \tilde{\theta} = E\theta + F\phi$$

$$\tilde{\phi} = S_T\theta + B\phi$$

(23)

where

$$B = -\frac{1}{\tau}A,\quad E = \frac{S_T + B}{D_T + A}A,\quad F = \frac{S_T + B}{D_T + A}D_T$$

and A is a positive root of the equation $A^2 + (\tau - 1)A - \tau\, S_T D_T = 0$.

The system of equations (17)-(22) upon using the transformation (23) assumes the following form:

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \frac{p}{\sigma}\right)w = R_T{}' \, a^2\theta - R_S{}' \, a^2\phi \tag{24}$$

$$\left(k_1\left(D^2 - a^2\right) - p\right)\theta = -w, \tag{25}$$

$$\left(k_2 (D^2 - a^2) - \frac{p}{\tau}\right)\phi = -\frac{w}{\tau} \tag{26}$$

with

$w = 0 = \theta = \phi = Dw$ at z=0 and z=1

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(27)

or

$$w = 0 = \theta = \phi = D^2 w \quad \text{at } z = 0 \text{ and } z = 1$$ (28)

or

$$w = 0 = \theta = \phi = Dw \quad \text{at} \quad z = 0$$
$$w = 0 = \theta = \phi = D^2 w \quad \text{at} \quad z = 1$$, (29)

where

$$k_1 = 1 + \frac{\tau\, D_T S_T}{A},\ k_2 = 1 - \frac{S_T D_T}{A}\ are\ positive$$
$$cons\tan ts$$
$$and\ R_T^{/} = \frac{(D_T + A)(R_T B + R_S S_T)}{BA - S_T D_T},$$
$$R_S^{/} = \frac{(S_T + B)(R_S A + R_T D_{T)}}{BA - S_T D_T}\ are\ \ respectively$$
$$the\ \mathrm{mod}ified\ thermal\ Rayleigh\ number\ and\ the$$
$$\mathrm{mod}ified\ concentration\ Rayleigh\ number.$$

Theorem2. If (p, w, θ, ϕ), $p = p_r + ip_i$ $p_r \geq 0, p_i \neq 0$ is a solution of equations (24)-(29), then $|p|^2 < R_S' \sigma$.

Proof: Since $p_i \neq 0$ we write equations (24)-(26) in the following convenient forms:

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \frac{p}{\sigma}\right)w - R_T{}^{/}\, a^2\theta +$$
$$+ R_S{}^{/}\ a^2\left[\frac{\tau k_2}{p}(D^2 - a^2)\phi + \frac{w}{p}\right] = 0, \qquad (30)$$

(28)

$$-R_{T'}' a^2\{(k_1(D^2 - a^2) - p)\theta + w\} = -w,$$ (31)

$$\frac{(D^2 - a^2)\tau^2 a^2 k_2}{p^*}\{\left(k_2(D^2 - a^2) - \frac{p}{\tau}\right)\phi + \frac{w}{\tau}\}$$
$$= -\frac{w}{\tau} \qquad (32)$$

Equation (1) reduces to the above equations with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\tau^2}{p*} R_S'\, a^2\, k_2^2 \end{bmatrix}$$

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B =

$$\begin{bmatrix} -\left(2a^2+\frac{p}{\sigma}\right) & 0 & \frac{\tau R_S'}{p}a^2k_2 \\ 0 & -R_T'a^2 & 0 \\ \frac{\tau R_S'}{p^*}a^2k_2 & 0 & -\frac{\tau R_S'}{p^*}a^2k_2 \\ & & \left(2a^2+\frac{p}{\sigma}\right) \end{bmatrix}$$

C =

$$\begin{bmatrix} a^4+\frac{pa^2}{\sigma}+\frac{R_S'}{p}a^2 & -R_T'{}^{a^2} & -\frac{\tau R_S'}{p}a^4k_2 \\ -R_T'a^2 & R_T'a^2(a^2k_1+p) & 0 \\ \frac{\tau R_S'}{p^*}a^4k_2 & 0 & \frac{\tau^2 R_S'}{p^*}a^2k_2 \\ & & (a^4k_2+\frac{pa^2}{\tau}) \end{bmatrix}$$

$$\chi\,(z) = \begin{pmatrix} w(z) \\ \theta(z) \\ \phi(z) \end{pmatrix}$$

Further, boundary conditions on χ confirms to those of $w, \theta\ and\ \phi$. Also

$$A_1 = p_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\tau^2}{|p|^2}R_S'\,a^2\,k_2^2 \end{bmatrix}$$

B_1 =

$$-p_i \begin{bmatrix} \frac{1}{\sigma} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2\ \tau^2}{|p|^2}R_S'\,a^2\,(a^2+\frac{p_r}{\tau})k_2^2 \end{bmatrix}$$

$C_1 =$

$$-p_i \begin{bmatrix} a^2\left(\frac{1}{\sigma}-\frac{R_S'}{|p|^2}\right) & 0 & 0 \\ 0 & -R_T'a^2 & 0 \\ 0 & 0 & -\frac{\tau^2}{|p|^2}R_S'\,a^4k_2 \\ & & (a^2k_2+\frac{2p_r}{\tau}) \end{bmatrix}$$

Now, with $= p_i$, conditions of Theorem 1 are satisfied and hence

$$|p|^2 < R_S'\sigma$$

This completes the proof of the Theorem.

(b) Double-Diffusive Convection Coupled with Cross-diffusions for Stern's type configuration

The governing equations and boundary conditions of this problem under Boussinesq approximation are given by equations (24) - (29) with

$$R_S' = -\widehat{R_S'}, \qquad R_T' = -\widehat{R_T'}$$

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where $R_S' > 0$, $R_T' > 0$ $(Stern\ [9])$.

Theorem 3: If (p, w, θ, ϕ), $p = p_r + ip_i$, $p_r \geq 0, p_i \neq 0$ is a solution of equations (24)-(29), then $|p|^2 < -R_T'\sigma$.

Proof: Since $p_i \neq 0$, we write governing equations for the present problem in the following convenient forms:

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \frac{p}{\sigma}\right)w + \hat{R}_T{}' \ a^2\left[\frac{k_1}{p}(D^2 - a^2)\theta + \frac{w}{p}\right] - \hat{R}_S' a^2 \phi = 0, \tag{33}$$

(28)

$$\hat{R}_T' \frac{a^2 k_1 (D^2 - a^2)}{p^*}\left[\left\langle k_1(D^2 - a^2) - p\right\rangle \theta + w\right] = 0 \tag{34}$$

$$-\tau\ \hat{R}_S' a^2\ \left[\ \left(k_2(D^2 - a^2) - \frac{p}{\tau}\right)\phi + \frac{w}{\tau}\ \right] = 0$$

(35)

Equation (1) reduces to the above equations with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\widehat{R_T'}}{P^*} a^2 k_1^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

B =

$$\begin{bmatrix} -\left(2a^2 + \frac{p}{\sigma}\right) & \frac{\widehat{R'_T}}{p} a^2 k_1 & 0 \\ \frac{\widehat{R_T'}}{p^*} a^2 k_1 & \widehat{R_T'} a^2 \frac{k_1}{p^*}(2a^2 k_1 + p) & 0 \\ 0 & 0 & -\tau \widehat{R_S'} a^2 k_2 \end{bmatrix}$$

C =

$$\begin{bmatrix} a^4 + \frac{pa^2}{\sigma} + \frac{\widehat{R'_T}}{p} a^2 & \frac{\widehat{R'_T}}{p} a^2 k_1 & -\widehat{R_S'}\, a^2 \\ -\frac{\widehat{R'_T}}{p^*} a^2\ k_1 & \frac{\widehat{R_T'}}{p^*} a^4 k_1 (a^2 k_1 + p) & 0 \\ -\widehat{R_S'}\, a^2 & 0 & \tau\ \widehat{R_S'} a^2 (a^2 k_2 + \frac{p}{\tau}) \end{bmatrix}$$

$$\chi\,(z) = \begin{pmatrix} w(z) \\ \theta(z) \\ \phi(z) \end{pmatrix}$$

Further, boundary conditions on χ confirms to those of $w, \theta\ and\ \phi$. Also

$$A_1 = p_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\widehat{R_T'}}{|p|^2} a^2 k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = -p_i \begin{bmatrix} \frac{1}{\sigma} & 0 & 0 \\ 0 & \frac{2\widehat{R_T'}}{|p|^2} a^2 k_1 \; (a^2 k_1 + p_r) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_1 = -p_i \begin{bmatrix} a^2\left(\frac{\widehat{R_T'}}{|p|^2} - \frac{1}{\sigma}\right) & 0 & 0 \\ 0 & \frac{\widehat{R_T'}}{|p|^2} a^4 k_1 \; (a^2 k_1 + 2p_r) & 0 \\ 0 & 0 & -\widehat{R_S'} a^2 \end{bmatrix}$$

Now, with $l = p_i$, conditions of Theorem 1 are satisfied and hence

$$|p|^2 < -R_T' \sigma.$$

This completes the proof of the Theorem.

REFERENCES

[1]. Fitts D.D. (1962), Nonequlibrium Thermodynamics: McGraw Hill, New York.
[2]. Groot S.R.de & Mazur. P. (1962), Nonequlibrium Thermodynamics, North Holland, Amsterdam.
[3]. Hurle D.T.J and. Jakeman E. (1971), Soret – driven thermosolutal convection, J. Fluid Mech., 47, 668-687.
[4]. Mohan H. (1998), Dufour - driven thermosolutal convection of the Veronis type, J. Math. Anal. Appl., 218, 569-580.
[5]. Warren F.W. (1976), On the method of Hermitian forms and its applications to some problems of hydrodynamic stability, Proc. Roy. Soc. London Ser. A 350, 213-218.
[6]. Chandersekher S. (1961), Hydrodynamic and Hydromagnetic Stability, Oxford Univ. Press, London.
[7]. Lin C.C. (1955), The Theory of Hydrodynamic Stability, Cambridge Univ. Press, Cambridge
[8]. Veronis G., (1965), On finite amplitude instability in thermohaline convection, J. Mar. Res., 23, 1-17
[9]. Stern M.E., (1960), The salt fountain and thermohaline convection, Tellus. 12, 172-175
[10]. Banerjee M.B,. Katoch D.C,. Dube G.S. & Banerjee K, (1981), Bounds for a growth rate of a perturbation in thermohaline convection, Proc. Roy. Soc. London, Ser. A 378, 301-304.
[11]. Neild D. (1967), The Thermohaline Rayliegh– Jaffreys Problems, J. Fluid Mech., 29, 545-558.
[12]. Krusin-Gutkowicz D., Collins M.A. & Ross J., (1979), Rayleigh-Benard instability in Non-Reactive Binary fluids, J. Phy. Fluids, 22, 1443-1460.